Quantum inconsistency of W_3-gravity models associated with magical Jordan algebras

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Abstract

It is shown that the Sugawara-type construction for W_3 algebra associated with the four magical Jordan algebras leads to the anomalous theory of W_3 gravity.

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The Virasoro algebra plays a key role in the study of string models and $2D$-conformal field theory, with various extensions involving Kac–Moody or superconformal generators. The W_3 algebra, proposed by Zamolodchikov [1] as a non-linear extension of the Virasoro algebra, has provided a basis for generalization of all $2D$-conformal field theory models to the case more wide and non-linear symmetry. In particular these are so-called the models of W_3 gravity [2, 3] and the W_3 strings [4, 5]. The W_3 Zamolodchikov's algebra underlying these models is described by the following operator products:

$$\begin{gathered}
\hbar^{-1}T(z)T(w) \sim \hbar\frac{d/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w};\\
\hbar^{-1}T(z)W(w) \sim \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w};\\
\hbar^{-1}W(z)W(w) \sim \hbar^2\frac{d/3}{(z-w)^6} + \hbar\frac{2T(w)}{(z-w)^4} + \hbar\frac{\partial T(w)}{(z-w)^3} + \\
+\hbar\frac{3}{10}\,\frac{\partial^2 T(w)}{(z-w)^2} + \hbar\frac{1}{15}\,\frac{\partial^3 T(w)}{z-w} + \frac{2\beta\Lambda(w)}{(z-w)^2} + \frac{\beta\partial\Lambda(w)}{z-w}.
\end{gathered} \tag{1}$$

The constant β is determined by the associativity requirement to take the value

$$\beta = \frac{16}{22+5d}, \tag{2}$$

and $\Lambda(z)$, which includes the non-linearities referred to above, is a composite field formed from the Virasoro generators [6]:

$$\Lambda(w) = \oint \frac{\mathrm{d}z}{z-w}T(z)T(w) - \frac{3}{10}\hbar\partial^2 T(w). \tag{3}$$

The general free-field ansatz for the W_3 generators in terms of the fundamental spin-one currents J^i reads [2]

$$\begin{aligned}
T &= -\,\frac{1}{4}g_{ij} :J^iJ^j: + \mathrm{i}\hbar a_j\partial J^j,\\
W &= -\,\frac{\mathrm{i}}{12}d_{ijk} :J^iJ^jJ^k: - \sqrt{\hbar}e_{ij} :J^i\partial J^j: + \mathrm{i}\hbar f_j\partial^2 J^j.
\end{aligned} \tag{4}$$

The notation $:\ :$ indicates normal ordering with respect to the modes of $J^i = \sum J^i_n z^{n-1}$ which have conventional operator products

$$J^i(z)J^j(w) \sim \frac{\hbar g^{ij}}{(z-w)^2}. \tag{5}$$

The leading terms in Eq. (4) directly correspond to the classical currents appearing in W_3 gravity.

In order to hold the W_3 algebra for the above realization T and W some set of conditions must be imposed on the structure tensors g_{ij}, d_{ijk}, e_{ij}, f_i, a_i.

In classical limit ($\hbar \to 0$) these conditions imply that d_{ijk} must necessarily be a structure constant of some Jordan algebra $\mathbb{J}_3$ with a cubic norm [7, 8]. The full set of constraints on the structure tensors as well as their solutions for the cases of generic Jordan algebras was formulated in ref. [9]. However the solution existence problem still remains unsolved for the other four cases known as the magical Jordan algebras $\mathbb{J}_3^{\mathbb{R}}$, $\mathbb{J}_3^{\mathbb{C}}$, $\mathbb{J}_3^{\mathbb{H}}$, $\mathbb{J}_3^{\mathbb{O}}$ with dimensions 5, 8, 14 and 26 respectively. The set of constraints on the structure tensors corresponding to the magical Jordan algebras takes the form

$$\begin{gathered} a_i = f_i = 0, \qquad g^{ij} d_{ijk} = 0; \\ d^m_{(ij} d_{k)ml} = \mu^2 g_{(ij} g_{kl)}; \\ e_{ij} = -e_{ji}, \qquad e_i{}^j e_{jk} = -\rho^2 g_{ik}; \\ e_i{}^l d_{jkl} = e_j{}^l d_{ikl}, \end{gathered} \tag{6}$$

where all indices are raised with g^{ij}-inverse matrix to g_{ij} and $i = 1, \ldots, n$, $n = 5, 8, 14, 26$,

$$\mu^2 = \frac{16}{22 + 5n}, \qquad \rho^2 = \frac{n - 2}{8(22 + 5n)}$$

In present paper we will prove the statement that the set of equations (6) has no solutions for any magical Jordan algebra. This fact makes impossible to construct consistent quantum theory of W_3 gravity models considered, as it is shown in the paper below.

First of all we note that making redefinition $d_{ijk} \to \frac{1}{\mu} d_{ijk}$, $e_{ij} \to \frac{1}{\rho} e_{ij}$ in equations (6) one can put $\rho = \mu = 1$. For purposes of following calculations it is convenient to introduce matrices

$$D(\mathbf{p}) = \| p^k d_k{}^i{}_j \|, \qquad M(\mathbf{q}, \mathbf{p}) = \| q^i p_j + p^i q_j \|, \qquad E = \| e_i{}^j \| \tag{7}$$

where $\mathbf{p}$ and $\mathbf{q}$ are arbitrary vectors. Then equations (6) can be rewritten in the matrix form

$$\begin{gathered} \mathrm{Sp}\ D(\mathbf{p}) = 0, \qquad E^{\mathrm{T}} = -E, \qquad E^2 = -\mathbb{I}; \\ \{D(\mathbf{p}), D(\mathbf{q})\} = -D(D(\mathbf{p})\mathbf{q}) + M(\mathbf{p}, \mathbf{q}) + (\mathbf{p}, \mathbf{q})\mathbb{I}; \\ ED(\mathbf{p}) = D(\mathbf{p})E^{\mathrm{T}}. \end{gathered} \tag{8}$$

Here brackets $\{\ ,\ \}$ stand for a matrix anticommutator, and $(\mathbf{p}, \mathbf{q}) = p_i q^i$ is inner product of two vectors.

It is easy to see, that matrices $D(\mathbf{r})$, $M(\mathbf{p}, \mathbf{q})$ and $\mathbb{I}$ generate a closed algebra with respect to their anticommutator, i.e. represent some Jordan algebra $\tilde{\mathbb{J}}$[1].

[1] It is interesting to note that the matrices $\{M\}$ generate an ideal of the $\tilde{\mathbb{J}}$, and associated factor-algebra $\tilde{\mathbb{J}}/\{M\}$ appears to be isomorphic to the initial $\mathbb{J}_3^{(\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O})}$.

Indeed

$$
\begin{gathered}
\{\mathbb{I}, M(\mathbf{p},\mathbf{q})\} = 2M(\mathbf{p},\mathbf{q}), \qquad \{\mathbb{I}, D(\mathbf{p})\} = 2D(\mathbf{p}), \\
\{D(\mathbf{r}), M(\mathbf{p},\mathbf{q})\} = M(D(\mathbf{r})\mathbf{p},\mathbf{q}) + M(\mathbf{p}, D(\mathbf{r})\mathbf{q}), \\
\{M(\mathbf{a},\mathbf{b}), M(\mathbf{p},\mathbf{q})\} = (\mathbf{a},\mathbf{p})M(\mathbf{b},\mathbf{q}) + (\mathbf{a},\mathbf{q})M(\mathbf{b},\mathbf{p}) + \\
+ (\mathbf{b},\mathbf{q})M(\mathbf{a},\mathbf{p}) + (\mathbf{b},\mathbf{p})M(\mathbf{a},\mathbf{q}).
\end{gathered} \tag{9}
$$

The above matrices have the following values of traces:

$$
\mathrm{Sp}\ D(\mathbf{p}) = 0, \qquad \mathrm{Sp}\ M(\mathbf{p},\mathbf{q}) = 2(\mathbf{p},\mathbf{q}), \qquad \mathrm{Sp}\ \mathbb{I} = n. \tag{10}
$$

The next matrix relation immediately follows from Eq. (8)

$$
D(\mathbf{p}) = -ED(\mathbf{p})E^{\mathrm{T}}. \tag{11}
$$

Razing to an odd power both side of Eq. (11) and computing traces of derived expressions we see

$$
\mathrm{Sp}\ (D(\mathbf{p}))^{2m+1} = 0, \quad m = 1, 2, \dots\ . \tag{12}
$$

On the other hand, using relation

$$
\mathrm{Sp}\ D^{n+1} = \frac{1}{2^n}\mathrm{Sp}\ \underbrace{\{\cdots\{\{}_{n} D, D\}, D\}, \cdots\} \tag{13}
$$

algebra (8), (9) and expressions for traces (10) one can sequentially obtain trace of any power of $D(\mathbf{p})$. For a few lower powers of the matrix $D(\mathbf{p})$ we get

$$
\begin{gathered}
\mathrm{Sp}\ D^2(\mathbf{p}) = \frac{2+n}{2}\mathbf{p}^2, \qquad \mathrm{Sp}\ D^4(\mathbf{p}) = \frac{10+3n}{8}(\mathbf{p}^2)^2, \\
\mathrm{Sp}\ D^3(\mathbf{p}) = \frac{2-n}{4}\mathbf{p}^3, \qquad \mathrm{Sp}\ D^5(\mathbf{p}) = \frac{2-5n}{16}\mathbf{p}^2\mathbf{p}^3, \\
p^2 \equiv p_i p^i, \qquad p^3 \equiv d_{ijk}p^i p^j p^k.
\end{gathered} \tag{14}
$$

Comparing Eqs. (14) with Eq. (12) we come to contradiction.

From the above discussion it appears that the classical W_3 algebra associated with magical Jordan algebras can not be extended to the quantum level by modifying classical currents by terms proportional to $\sqrt{\hbar}$ and $\hbar$. The quantum selfconsistency is known to require the BRST charge operator to be nilpotent. The quantum BRST charge for Zamolodchikov algebra has the form [10, 11]

$$
\Omega = \oint \mathrm{d}z : \{c(T - \bar{c}c' - \frac{1}{2}\bar{c}'c - 3\bar{b}b' - 2\bar{b}'b) - bW - \beta b\bar{c}b'T - \frac{5d}{1044}(\frac{1}{15}\bar{c}''b'b - \frac{1}{6}\bar{c}b''b')\} : . \tag{15}
$$

Here $c(z)$ and $\bar{c}(z)$ are a pair of conformal ghosts, while $b(z)$ and $\bar{b}(z)$ are a new pair of ghosts related with W. The requirement of nilpotency $\Omega^2 = 0$ fixes the

critical value of $d = 100$. Since the algebra of W_3 symmetry is unclosed in the quantum level the Ω charge is not nilpotent and this as well known gives rise to anomalies. Our result supplements the findings of ref. [9] and makes possible to assert that all anomaly-free models of W_3 gravity one–to–one correspond to the generic Jordan algebras $\mathbb{J}_3$.

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References

1. A.B. Zamolodchikov, Teor. Mat. Fiz., **65**, 374 (1985).
2. C.M. Hull, Phys. Lett. B, **240**, 110 (1990).
3. K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Phys. Lett. B, **243**, 245 (1990).
4. C.N. Pope, L.J. Romans and K.S. Stelle, Phys. Lett. B, **269**, 287 (1991).
5. C.N. Pope, L.J. Romans, E. Sezgin and K.S. Stelle, Phys. Lett. B, **274**, 298 (1992).
6. B.L. Feigin and D.B. Fuchs, Funktc. Anal. Pril., **16**, 47 (1982).
7. C.M. Hull, Nucl. Phys. B, **353**, 707 (1991).
8. M. Günuydin, G. Sierra and P.K. Townsend, Nucl. Phys. B, **242**, 244 (1984).
9. L.J. Romans, Nucl. Phys. B, **352**, 829 (1991).
10. J. Thierry-Mieg, Phys. Lett. B, **197**, 368 (1987).
11. Won-Sang Chung and Jae-Kwan Kim, Phys. Rev. D, **41**, 1336 (1990).